

Statistical Mechanical Models and Topological Color Codes

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We find that the overlapping of a topological quantum color code state, representing a quantum memory, with a factorized state of qubits can be written as the partition function of a 3-body classical Ising model on triangular or Union Jack lattices. This mapping allows us to test that different computational capabilities of color codes correspond to qualitatively different universality classes of their associated classical spin models. By generalizing these statistical mechanical models for arbitrary inhomogeneous and complex couplings, it is possible to study a measurement-based quantum computation with a color code state and we find that their classical simulatability remains an open problem. We complement the measurement-based computation with the construction of a cluster state that yields the topological color code and this also gives the possibility to represent statistical models with external magnetic fields.

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I. INTRODUCTION

Recently, a very fruitful relationship has been established between partition functions of classical spin models and a certain class of quantum stabilizer states with topological protection [1], [2]. The topological quantum code states considered so far in these studies correspond to the toric code states introduced by Kitaev [3], [4]. The classical spin model that emerges when a planar toric code is projected onto a product state of single-qubits with very specific coefficients is the standard classical Ising model in two dimensions with homogeneous real couplings and zero magnetic field.

Single-qubit measurements also appear naturally in a measurement-based computation (MQC) scheme [5], [6]. Thus, these connections between classical spin models and topological quantum states are also useful to test whether those topological states are efficiently classically simulable with MQC. It has been shown that MQC with a planar Kitaev code state as input can be efficiently simulated in a classical computer if at each step of the computation, the sets of measured qubits form simply connected subsets of the two-dimensional lattice [2]. The connection of classical spin models with measurement-based quantum computation has been shown to be useful to prove the completeness of the classical 2D Ising model with suitably tuned complex nearest-neighbor couplings in order to represent the partition function of the classical Ising model on arbitrary lattices, with inhomogeneous pairwise interactions and local magnetic fields [7].

Topological color codes (TCC) were introduced to implement the set of quantum unitary gates of the whole Clifford group by means of a topological stabilizer code in a two dimensional lattice [8], and then generalized to three dimensional lattices in order to achieve a universal set of topological quantum gates [9]. These 2D and 3D realizations of TCC are instances of general D -dimensional realizations. We call those lattices related to these codes as D -colexes (for color complexes), and they are D -dimensional lattices with coordination number $D + 1$ and certain colorability properties. Moreover,

this codes can also appear as the ground state of suitable Hamiltonians, and the corresponding quantum systems are brane-net condensates. [10].

Given these nice properties exhibited by the topological color codes, it is natural to ask what type of classical spin models can be constructed out of them and see whether they belong or not to the same universality class of the classical Ising model arising in the Kitaev model. In this work we address this issue and find that the overlapping of a TCC state with a product state of single qubits with appropriate coefficients is mapped onto the partition function of the 3-body classical Ising model on the dual lattice of the original lattice where the color code is defined. For concreteness, we consider the triangular and the ‘Union Jack’ lattice for these classical many-body spin systems. This represents a sharp difference with the result obtained with the topological states in the Kitaev code. In fact, the universality classes of the 3-body classical Ising model in several lattices are quite different from the corresponding universality class of the standard 2-body Ising model.

Moreover, we also study the topological color code states in a MQC scenario to test their classical simulability. We find that the current state of knowledge in statistical mechanical models with 3-body interactions, arbitrary inhomogeneous complex couplings and lattice shapes is much less developed than the 2-body Ising model which is relevant for the case of the toric code states. Thus, we conclude that the classical simulability of TCC states with MQC remains an open problem.

In a MQC, the usual initial many-particle entangled state is a cluster state [5], [6] instead of a topological code. Then, we also show how to construct a color code state from a certain cluster state. Interestingly enough, this construction turns out to be useful for the description of statistical mechanical systems with 3-body interactions in the presence of an external field.

This paper is organized as follows: in Sect.II we give an introduction to the topological color code states needed to present in Sect.III the mapping onto the classical 3-body Ising model in the triangular and Union Jack lat-

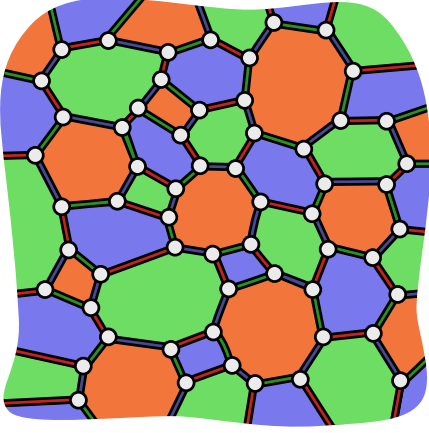


Figure 1: An example of 2-colex. Both edges and face are 3-colorable and they are colored in such a way that green edges connect green faces and so on and so forth for red and blue edges and faces.

tices. In Sect.IV we study the measurement-based quantum computation with topological color code states by generalizing the results of the previous section. In Sect.V we show how to prepare a topological color code from a cluster state as those introduced in MQC. This is also useful for studying partition functions of statistical mechanical models with 3-body interactions and external magnetic fields. Sect.VI is devoted to conclusions.

II. TOPOLOGICAL COLOR CODES

A. Construction

Let us start by recalling the notion of a Topological Color Code (TCC) in order to see what type of classical spin models we obtain from them with appropriate projections onto factorized quantum states and specific lattices.

A TCC, denoted by \mathcal{C} , is a quantum stabilizer error correction code constructed with certain class of two-dimensional lattices called 2-colexes. The word colex is a contraction that stands for color complex, where complex is the mathematical terminology for a rather general lattice [11]. A 2-colex, denoted by \mathfrak{C}_2 , is a 2D trivalent lattice which has 3-colorable faces and is embedded in a compact surface of arbitrary topology like a torus of genus g . A trivalent lattice is one for which three edges meet at every vertex. The property of being 3-colorable means that the faces (or plaquettes) of the lattice can be colored with these colors in such a way that neighboring faces never have the same color. We select as colors red (r), green (g) and blue (b). An example of a 2-colex construction is shown in Fig. 1.

Edges can be colored in according to the coloring of the faces. In particular, we attach red color to the edges

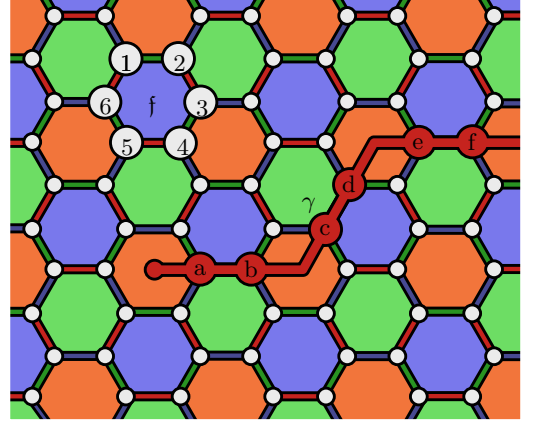


Figure 2: An hexagonal lattice is an instance of 2-colex. Numbered vertices belong to the face f . Vertices labeled with letters correspond to the red string γ displayed. γ is an open string, because it has an endpoint in a red face.

that connect red faces, and so on and so forth for the blue and green edges/faces. When studying higher dimensional colexes, it turns out that the coloring of the edges is the key property of D -colexes: all the information about a D -colex is encoded in its 1-skeleton, i.e., the set of edges with its coloring[10].

Given a 2-colex \mathfrak{C}_2 , a TCC \mathcal{C} is constructed by placing one qubit at each vertex of the colored lattice. Let us denote by \mathfrak{V} , \mathfrak{E} and \mathfrak{F} the sets of vertices \mathfrak{v} , edges \mathfrak{e} and faces \mathfrak{f} , respectively, of the given 2-colex. Then, the generators of the stabilizer group, denoted by \mathcal{S} , are given by face operators only. For each face \mathfrak{f} , they come into two types depending whether they are constructed with Pauli operators of X - or Z -type:

$$\begin{aligned} X_{\mathfrak{f}} &:= \bigotimes_{\mathfrak{v} \in \mathfrak{f}} X_{\mathfrak{v}}, \\ Z_{\mathfrak{f}} &:= \bigotimes_{\mathfrak{v} \in \mathfrak{f}} Z_{\mathfrak{v}}, \end{aligned} \quad (1)$$

and there are no generators associated to lattice vertices. For example, an hexagonal lattice is an instance of a 2-colex, see Fig. 2. The operators for the face \mathfrak{f} displayed in the figure take the form $X_{\mathfrak{f}} = X_1 X_2 X_3 X_4 X_5 X_6$, $Z_{\mathfrak{f}} = Z_1 Z_2 Z_3 Z_4 Z_5 Z_6$. A given state $|\Psi_c\rangle \in \mathcal{C}$ is left trivially invariant under the action of the face operators,

$$X_{\mathfrak{f}}|\Psi_c\rangle = |\Psi_c\rangle, \quad Z_{\mathfrak{f}}|\Psi_c\rangle = |\Psi_c\rangle, \quad \forall \mathfrak{f} \in \mathfrak{F}. \quad (2)$$

An erroneous state $|\Psi\rangle_e$ is one that violates conditions (2) for some set of face operators of either type. As the generator operators $X_{\mathfrak{f}}, Z_{\mathfrak{f}} \in \mathcal{S}$ satisfy that they square to the identity operator, $(X_{\mathfrak{f}})^2 = \mathbb{I} = (Z_{\mathfrak{f}})^2, \forall \mathfrak{f} \in \mathfrak{F}$, then an erroneous state is detected by having a negative eigenvalue with respect to some set of stabilizer generators: $X_{\mathfrak{f}}|\Psi\rangle_e = -|\Psi\rangle_e$ and/or $Z_{\mathfrak{f}}|\Psi\rangle_e = -|\Psi\rangle_e$.

Interestingly enough, it is possible to construct a quantum lattice Hamiltonian H_c such that its ground state is

degenerate and corresponds to the TCC \mathcal{C} , while the erroneous states are given by the spectrum of excitations of the Hamiltonian [8]. Such Hamiltonian is constructed out of the generators of the topological stabilizer group \mathcal{S} ,

$$H_c = - \sum_{f \in \mathfrak{F}} (X_f + Z_f). \quad (3)$$

The ground state of this Hamiltonian exhibits what is called a topological order [12], as opposed to a more standard order based on an spontaneous symmetry breaking mechanism. One of the signatures of that topological order is precisely the topological origin of the ground state degeneracy: the number of degenerate ground states depends on topological invariants like Betti numbers [10]. In two dimensional lattices, the relevant Betti number corresponds to the Euler characteristic χ of the surface where the 2-colex is embedded.

B. String-net operators

In order to better understand both the ground state and excitations of this Hamiltonian and their topological properties, it is rather convenient to introduce the set of string operators that can be defined on a 2-colex \mathfrak{C}_2 . String operators are generalizations of face operators (1) that can be either open or closed, i.e., with or without end-points. These strings are topological and like in Kitaev model, the homology is defined on \mathbf{Z}_2 since we work with two-level quantum systems located at the sites of the lattice. However, in a TCC we have an additional ingredient to play around: color. Let us split the sets of edges and faces into colored subsets denoted by $\mathfrak{E} := \mathfrak{E}_r \cup \mathfrak{E}_g \cup \mathfrak{E}_b$ and $\mathfrak{F} := \mathfrak{F}_r \cup \mathfrak{F}_g \cup \mathfrak{F}_b$, where \mathfrak{E}_r is the subset of red edges, and similarly for the rest of subsets.

A colored string γ is a collection of edges of a given color. Thus, a blue string γ takes the form $\gamma = \{e_i\}$ with $e_i \in \mathfrak{E}_b$. The definition of colored string operators is completely analogous to that of face operators:

$$X_\gamma := \bigotimes_{e \in \gamma} X_e, \quad Z_\gamma := \bigotimes_{e \in \gamma} Z_e, \quad (4)$$

where, in turn, $X_e = X_{v_1} \otimes X_{v_2}$ if v_1 and v_2 are the sites at the ends of the edge e , and similarly for $Z_e = Z_{v_1} \otimes Z_{v_2}$. For instance, consider the red string operator in Fig. 2, where we have $X_\gamma = X_a X_b X_c X_d X_e X_f \dots$, $Z_\gamma = Z_a Z_b Z_c Z_d Z_e Z_f \dots$.

Colored strings are open if they have endpoints. These endpoints are localized at faces which share color with the string. In particular, a face f is an endpoint of γ if the number of edges of γ meeting at f is odd, see Fig. 2. In terms of string operators, a face f is an endpoint of γ if $\{X_\gamma, Z_f\} \neq 0$ or, equivalently, if $\{Z_\gamma, X_f\} \neq 0$. Thus, open string operators do not commute with those face operators in their ends. In other words, a string operator that commutes with all the face operators must

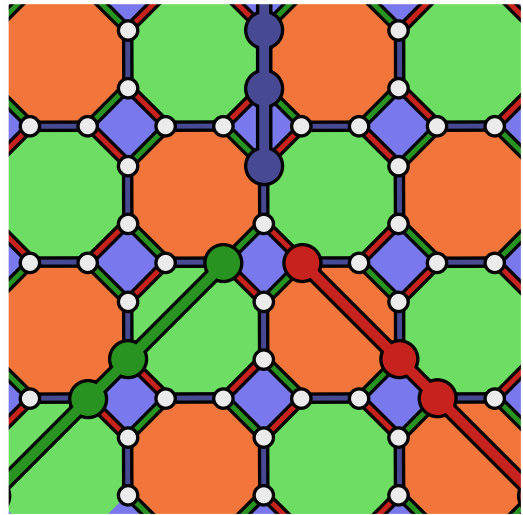


Figure 3: A 4-8 lattice is an instance of 2-colex. A closed string-net is displayed, composed of 3 strings of different colors meeting at a branching point. The string-net is closed because at each face we find an even number of its vertices.

correspond to a closed string, that is, a string without endpoints. In terms of the Hamiltonian (3), string operators produce quasi-particle excitations at their ends when applied to the ground state. These quasi-particle excitations are Abelian anyons.

Closed strings are mainly of two types. They can be homologically trivial, meaning that they are the boundary of certain area of the surface, or homologically non-trivial. In terms of operators, a string is a boundary iff its string operators belong to the stabilizer group \mathcal{S} of the color code \mathcal{C} . Such boundary string operators are thus products of face operators. In fact, face operators themselves are the basic boundary string operators.

Although we have introduced colored strings and the corresponding operators for illustrative purposes, in fact in a TCC we have to deal with more general objects, namely string-nets. A string-net is a collection of strings meeting at certain branching or ramification points. An example of these types of configurations are shown in Fig. 3.

A string-net γ is a collection of vertices $\gamma \subset \mathfrak{V}$. Equivalently, γ is a formal sum of lattice vertices $v \in \mathfrak{V}$ with coefficients $\gamma_v \in \mathbf{Z}_2$, i.e.,

$$\gamma = \sum_{v \in \mathfrak{V}} \gamma_v v, \quad (5)$$

where $\gamma_v = 1$ if $v \in \gamma$ and $\gamma_v = 0$ otherwise. Given a string-net γ , we define the string-net operators

$$X_\gamma := \bigotimes_v X^{v_\gamma}, \quad Z_\gamma := \bigotimes_v Z^{v_\gamma} \quad (6)$$

Just as in the case of colored strings, we can talk about open and closed string-nets, and about trivial and non-trivial closed string-nets. In terms of operators, the situation is exactly the same as with strings. That is, a

string-net γ has an endpoint at a face f if $\{X_\gamma, Z_f\} = 0$, it is closed if its string-net operators commute with all the face operators, and it is a boundary if it is a product of face operators. In order to translate these ideas into purely geometric terms, we can define the boundary operator

$$\partial_c \gamma := \sum_f x_f f, \quad x_f = \begin{cases} 0, & |\gamma \cap f| \text{ is even,} \\ 1, & |\gamma \cap f| \text{ is odd,} \end{cases} \quad (7)$$

where $|\gamma \cap f|$ is the number of vertices that γ and f share. Thus $\partial_c \gamma$ is the formal sum of the endpoints of γ . It is also natural to define an operator ∂_c for faces

$$\partial_c f := \sum_v x_v v, \quad x_v = \begin{cases} 0, & v \notin f, \\ 1, & v \in f, \end{cases} \quad (8)$$

so that $\partial_c f$ is the string-net composed of the vertices of f . With this definitions, γ is closed if and only if

$$\partial_c \gamma = 0, \quad (9)$$

and it is a boundary if and only if there exist a collection of faces $S = \sum_f S_f f$ such that

$$\gamma = \partial_c S. \quad (10)$$

It is possible to give explicit expressions for the states of the TCC or, equivalently, for the ground states of the Hamiltonian (3). The states are superpositions of all possible closed string-nets, a typical feature of the ground states of systems with topological order [12], [13]. The following is an un-normalized ground state for any given 2-colex [8], [10]

$$\begin{aligned} |\Psi_c\rangle &:= \prod_f (1 + X_f) |0\rangle^{\otimes |\mathfrak{V}|} \\ &= \sum_{\gamma \in \Gamma_0} X_\gamma |0\rangle^{\otimes |\mathfrak{V}|} =: \sum_{\gamma \in \Gamma_0} |\gamma\rangle, \end{aligned} \quad (11)$$

where $|\mathfrak{V}|$ is the number of vertices in the 2-colex \mathfrak{C}_2 , Γ_0 denotes the set of boundary string-nets and $|0\rangle$ is the eigenstate $Z|0\rangle = |0\rangle$.

The degeneracy of the ground state or, equivalently, the number of logical states encoded in the color code, depends on the topology of the lattice. For a general 2-colex \mathfrak{C}_2 with Euler characteristic $\chi(\mathfrak{C}_2) := |\mathfrak{V}| - |\mathfrak{E}| + |\mathfrak{F}|$, the number k of encoded qubits is given by $k = 4 - 2\chi(\mathfrak{C}_2) := 2h_1$ [8], where h_1 is the first Betti number of the surface where the 2-colex is embedded [10]. These additional ground states can be obtained from the one given by (11) by the action of the encoded logical operators \bar{X}_i, \bar{Z}_i with $i = 1, \dots, k$. These, in turn, take the form of string-net operators of non-trivial closed string-nets [8].

For the purpose of this work, we shall be interested only in a representative ground state like (11). Thus, we will have to consider suitable surface topologies such that the corresponding TCC is unique. We will return over this issue later when we consider particular lattices.

III. CONNECTION WITH CLASSICAL SPIN SYSTEMS

A. Overlap and Partition Function

Now, we come to the issue of what type of classical spin models may arise from the color code state (11) when we project it onto a product state of a number of qubits given by $|\mathfrak{V}|$. In this section we shall not consider the most general factorized state, but one specifically adapted for the purpose of this connection in its most simple form, namely,

$$|\Phi_P\rangle := \bigotimes_{v \in \mathfrak{V}} |\phi\rangle_v; \quad |\phi\rangle_v := \cosh(\beta J) |0\rangle_v + \sinh(\beta J) |1\rangle_v, \quad (12)$$

with $\beta := 1/k_B T$ the inverse temperature parameter.

The classical spin model arises when computing the overlapping between the ground state of the color code Hamiltonian (11) and this factorized state (12),

$$O(\beta J) := \langle \Psi_c | \Phi_P \rangle. \quad (13)$$

Using (11) and (12) we get the following expression for this overlapping,

$$O(\beta J) = \sum_{\gamma \in \Gamma_0} \langle \gamma | \bigotimes_{v \in \mathfrak{V}} |\phi\rangle_v = (\cosh(\beta J))^{|\mathfrak{V}|} \sum_{\gamma \in \Gamma_0} u^{|\gamma|}, \quad (14)$$

$u := \tanh(\beta J)$ and $|\gamma|$ is the number of vertices of γ .

We want to relate (14) to the partition function of a classical spin system. So let \mathfrak{C}_2 be an arbitrary 2-colex. Consider the dual lattice Λ . The vertices of Λ correspond to the faces of \mathfrak{C}_2 , and the faces of Λ are vertices in \mathfrak{C}_2 . In particular, Λ is a lattice in which all faces are triangular and vertices are 3-colorable. Moreover, for any such lattice Λ there exist a suitable dual 2-colex \mathfrak{C}_2 .

So let us associate a classical system to Λ by attaching classical spin variables $\sigma_i = \pm 1$ to each of its sites i (equivalently, to each face f of \mathfrak{C}_2). The classical Hamiltonian is

$$\mathcal{H} := -J \sum_{\langle i,j,k \rangle} \sigma_i \sigma_j \sigma_k, \quad (15)$$

where J is a coupling constant and the sum $\sum_{\langle i,j,k \rangle}$ is over all triangles with spins $\sigma_i \sigma_j \sigma_k$ at their vertices. Thus, we have a classical Ising model with 3-body interactions. The case $J > 0$ corresponds to a ferromagnetic model with an even parity to be discussed below, and similarly $J < 0$ to an antiferromagnetic model with odd parity. The partition function of the model is

$$\mathcal{Z}(\beta J) := \sum_{\{\sigma\}} e^{\beta J \sum_{\langle i,j,k \rangle} \sigma_i \sigma_j \sigma_k}, \quad (16)$$

where the sum $\sum_{\{\sigma\}}$ is over all possible configurations of spins. The point then is that we have

$$\mathcal{Z}(\beta J) = 2^N O(\beta J), \quad (17)$$

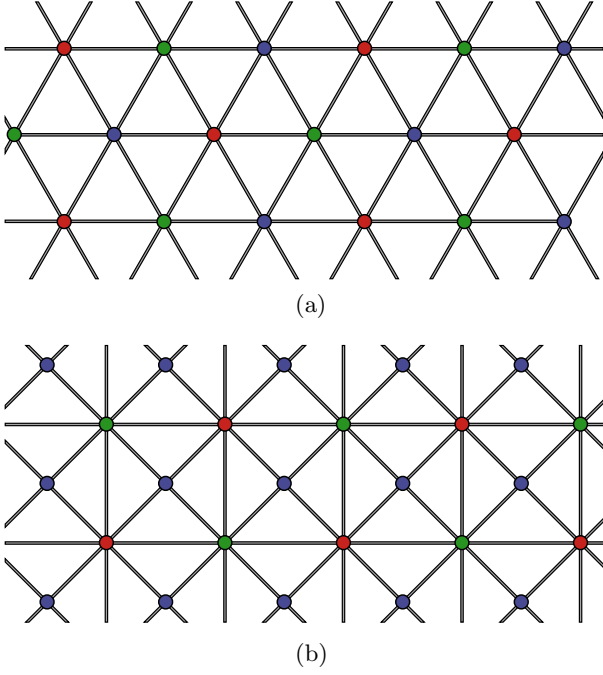


Figure 4: Two instances of dual lattices of a 2-colex, which have triangles as faces and have 3-colorable sites. The triangular lattice (a) is dual to the hexagonal one. The Union Jack lattice (b) is dual to the square-octogonal or 4-8 lattice.

where N is the number of sites.

Before we show why this identity holds, let us give a pair of representative examples of dual lattices \mathfrak{C}_2 and Λ . First, if the 2-colex is an hexagonal lattice then the dual lattice Λ is a *triangular lattice*, Fig. 4(a). Second, if the 2-colex is a square-octogonal lattice (also denoted by 4-8 lattice), then its dual is a *Union Jack lattice*, see Fig. 4(b). The relevance of these examples is two-fold. On the one hand, the hexagonal lattice is the simplest lattice for a 2-colex and the 4-8 lattice is the simplest one when we want to obtain TCC with certain transversality properties for quantum computation (see below). On the other hand, 3-body classical Ising-models on both lattices have been studied in statistical mechanics to some extent.

To prove (17), let us start expanding the partition function $\mathcal{Z}(\beta J)$ (16) using the following identity,

$$e^{\beta J \sigma_i \sigma_j \sigma_k} = \cosh(\beta J) + \sigma_i \sigma_j \sigma_k \sinh(\beta J). \quad (18)$$

Inserting it into (16), we may expand the partition function as

$$\mathcal{Z}(\beta J) = (\cosh(\beta J))^N \sum_{\{\sigma\}} \prod_{\langle i,j,k \rangle} (1 + u \sigma_i \sigma_j \sigma_k). \quad (19)$$

Let us rewrite (45) in the form

$$\mathcal{Z}(\beta J) = (\cosh(\beta J))^N \sum_{\delta} u^{|\delta|} \sum_{\{\sigma\}} \prod_{\langle i,j,k \rangle} (\sigma_i \sigma_j \sigma_k)^{\delta_{ijk}}, \quad (20)$$

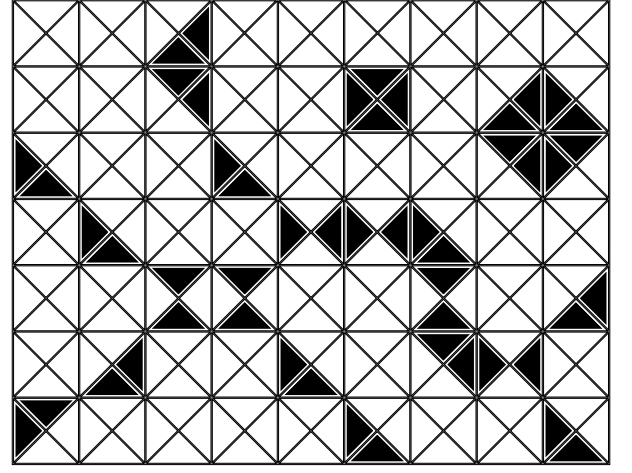


Figure 5: A typical chain of triangles $\delta \in \Delta_0$ in a triangular lattice. It is understood that only part of the lattice is displayed. Black triangles represent the elements of δ . The fact that $\delta \in \Delta_0$ means that at each vertex always meet an even number of triangles.

where $\delta = \sum_{\langle i,j,k \rangle} \delta_{ijk} \triangle_{ijk}$ is a chain of triangles, that is, a formal sum over triangles with binary coefficients, and $|\delta|$ is the number of triangles in δ . Using the identities

$$\sum_{\sigma=\pm 1} \sigma^{n_o} = 0, \quad \sum_{\sigma=\pm 1} \sigma^{n_e} = 2, \quad (21)$$

where n_o and n_e are odd and even numbers, respectively, we get

$$\mathcal{Z}(\beta J) = (2 \cosh(\beta J))^N \sum_{\delta \in \Delta_0} u^{|\delta|}, \quad (22)$$

where Δ_0 contains those chains of triangles such that at any given site i an even number of triangles meet, as shown in Fig. 5. In fact, this type of expansion is called a high-temperature expansion of the partition function of a statistical mechanical model [16].

In order to compare (40) and (52), we simply observe that triangles \triangle_{ijk} in Λ correspond to vertices of the 2-colex $\mathfrak{v} \in \mathfrak{V}$. This correspondence relates in an obvious way a string-net γ with a triangle chain δ , in such a way that Δ_0 is identified with Γ_0 . Therefore, we have the desired relationship between the overlapping and the partition function (17).

Although the previous derivation was performed for a model with uniform couplings, it is possible to obtain a completely analogous result for triangle dependent couplings J_{ijk} . For simplicity we have preferred to do the exposition with uniform couplings because, in fact, the case of non-uniform couplings is contained in the more general case of non-uniform couplings with non-uniform external field, to be considered in section V.

B. Consequences

We hereby draw a series of very important consequences from these results, that will continue in the next section.

i/ Interactions: We see that there is a clear qualitative difference between topological color code states and Kitaev's toric code states since they yield quite different type of spin interactions: a TCC state yields a 3-body interaction like in (16), while a Kitaev's code produces the standard 2-body classical Ising model, namely,

$$\mathcal{Z}_{\text{Ising}}(\beta J) := \sum_{\{\sigma\}} e^{\beta J \sum_{\langle i,j \rangle} \sigma_i \sigma_j}. \quad (23)$$

ii/ Symmetry: A distinctive feature of our result (16) is that the classical spin model associated to the TCC state does not possess the up-down \mathbf{Z}_2 spin-reversal symmetry. However, the partition function (16) exhibits a $\mathbf{Z}_2 \times \mathbf{Z}_2$ symmetry. Recall that the lattice Λ is 3-colorable at sites, so that we can redundantly label our classical spin variables as σ_i^c with $c = r, g, b$ the color at site i . Then the change of variables

$$\sigma_i^c \rightarrow s(c) \sigma_i^c, \quad s(r)s(g)s(b) = 1, s(c) = \pm 1, \quad (24)$$

gives a global symmetry, which has symmetry group $\mathbf{Z}_2 \times \mathbf{Z}_2$ because $s(b) = s(r)s(g)$. The ground states have to display this symmetry, in fact. Consider states in which the values of the spin variables only depend on the color, that is, for which $\sigma_i^c = f_c$ with $f_c = \pm 1$. Such states can be labelled with the tag (f_r, f_g, f_b) . Then it is easy to check that for the ferromagnetic case $J > 0$ the ground states are labelled with the positive parity tags $(+, +, +), (+, -, -), (-, +, -), (-, -, +)$ whereas in the antiferromagnetic case $J < 0$ they are labelled with the negative parity tags $(-, -, -), (-, +, +), (+, -, +), (+, +, -)$. Thus, each parity sector, or classical ground state of (15) is fourfold degenerate. Notice that the 3-body Ising model in such 3-colorable lattices of triangles shows no frustration, as opposed to the standard Ising model (23) in such lattices which is indeed frustrated.

Remarkably, the gauge group underlying the topological order related to Hamiltonian (3) is also $\mathbf{Z}_2 \times \mathbf{Z}_2$. The situation is the same also with toric codes, where the global symmetry of the classical system is \mathbf{Z}_2 and the gauge group for the toric code topological order is \mathbf{Z}_2 . This is certainly not a matter of chance, since one can relate the types of domain walls in the classical system to the types of condensed strings in the quantum system.

iii/ Selfduality: The models in the triangular and Union Jack lattices turn out to be self-dual like the usual 2-body Ising model, with a critical temperature β_c given by the same condition,

$$\sinh 2K_c = 1, \quad K_c := \beta_c J_c = 0.4407. \quad (25)$$

Duality is a property between high-temperature and low-temperature expansions of a statistical mechanical

model like (16) or (23). A high-temperature expansion is a polynomial in the variable $u = \tanh(\beta J)$ that is small when $T \rightarrow \infty$, while a low temperature expansion is another polynomial in the variable $u^* := e^{-2\beta J^*}$ that is small in the limit $T \rightarrow 0$. Then, a self-duality is a relationship between the high-temperature expansion of one classical spin model in a given lattice Λ and the low-temperature expansion of the same lattice. This is precisely the case of the 3-body Ising model (16) on both the triangular and Union Jack lattices [21], [22] and the standard Ising model (23) [20].

iv/ Universality Classes: Interestingly enough, the 3-body Ising model on the triangular lattice [17], [18] and the Union-Jack lattice [19] are exactly solvable models under certain circumstances and this fact allows us to check their criticality properties when compared with those of the standard Ising model solution.

The critical exponent for the specific heat in the 3-body Ising model on the triangular lattice is $\alpha = \frac{2}{3}$. This represents a power law behaviour which is in sharp contrast with the well-known logarithmic divergence ($\alpha = 0$) of the specific heat in the standard Ising model (23). Other representative exponents are also different: the correlation length exponent is $\nu = \frac{2}{3}$ (vs. $\nu = 1$), the magnetization exponent is $\beta = \frac{1}{12}$ (vs. $\beta = \frac{1}{8}$), while they share the same two-point correlation function exponent at the critical point $\eta = \frac{1}{4}$.

For the 3-body Ising model on the Union Jack lattice, the specific heat critical exponent is also remarkably different $\alpha = \frac{1}{2}$. In fact, if the coupling constant J is allowed to be anisotropic, then even the critical exponent α may take on a set of continuous values in $(0, \frac{1}{2})$ depending on a parameter related to the coupling constants [19].

The computational capabilities of a topological color code depends on the 2-colex lattices where it is defined. For a TCC on a square-octagonal lattice it is possible to implement the whole Clifford group of unitary gates generated by the set of gates $\{H, K^{\frac{1}{2}}, \Lambda_2\}$, where H is the Hadamard gate, $K^{\frac{1}{2}}$ the $\pi/8$ -gate and Λ_2 the CNOT gate [8]. However, for a 2-colex like the hexagonal lattice the set of available gates is more reduced since the $\pi/8$ -gate cannot be implemented topologically [8]. Thus we point out a remarkable connection between different computational capabilities of color codes that correspond to qualitatively different universality classes of their associated classical spin models, despite the fact that both color codes have the same topological order.

C. Borders

If we want to consider classical systems of spins with a finite number of sites, then we have to introduce either borders or a nontrivial topology. Since in TCCs the nontrivial topology gives rise to degeneracy, it is preferable to have borders. Also, borders play a role for the ideas to be explained in the next section.

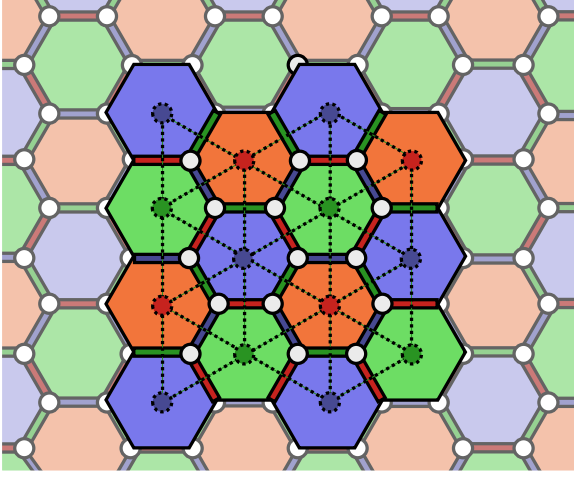


Figure 6: Here both an hexagonal 2-colex \mathfrak{C}_2 and its dual triangular lattice Λ (dashed) are displayed to illustrate how borders are introduced in the 2-colex if Λ has borders. All vertices of the 2-colex which are not triangles in Λ have been removed, and also all the faces which keep no vertices. The faces that only keep part of their vertices remain, but only their Z face operators are kept in the stabilizer.

In TCCs, borders can be of several types. For example, in [8] it was shown how to build borders of a given color. Here our guide to construct the border must be the dual lattice Λ , which now has a border, along with the properties of the classical system. Then, as shown in Fig. 6, in order to construct the stabilizer for a TCC with border in such a way that (17) remains true is to start with an infinite 2-colex \mathfrak{C}_2 and then keep only part of it following certain criteria. (i) Keep those vertices \mathbf{v} of \mathfrak{C}_2 which correspond to triangles in Λ . (ii) For those faces \mathbf{f} of \mathfrak{C}_2 which keep all their vertices, we keep the face operators $X_{\mathbf{f}}$ and $Z_{\mathbf{f}}$. (iii) For those faces \mathbf{f} which only keep a subset \mathbf{f}' of their vertices, we introduce a face operator $Z_{\mathbf{f}'}$ acting on those qubits. Condition (i) ensures the correspondence between triangle chains in Λ and string-nets in \mathfrak{C}_2 . Conditions (ii) and (iii) ensure the correspondence between Γ_0 and Δ_0 .

IV. MEASUREMENT-BASED QUANTUM COMPUTATION WITH COLOR CODES

In a measurement-based quantum computer (MCQ) [5], information processing is carried out via a sequence of one-qubit measurements on an initialized entangled quantum register. This is an alternative to the standard gate-based quantum computation that can simulate quantum networks efficiently.

An interesting problem is to study the performance of the MQC when the initial entangled multiparticle state is a topological toric code like in Kitaev's model [2]. In particular, under which general circumstances the MQC

based on the planar Kitaev code can be efficiently simulated by a classical computer. The answer to this question is that the planar code state can be efficiently simulated on a classical computer if at each step of MQC, the sets of measured and unmeasured qubits correspond to simply connected subsets of the lattice [2].

Likewise, another very interesting problem is whether a topological color code state is classically simulable in an scenario of measurement-based quantum computation (MQC). By extending the results of sect. III, it is possible to address this problem here. Our aim is to see what conclusions can we learn from the statistical mechanical models in order to test the classical simulability by MQC of the topological code states.

The results of sect. III can be interpreted as a complete projective measurement of the planar topological color code state (11) onto a very specific product of single qubit states (12). This type of global measurements are not enough for doing a MQC based on the color code state. Instead, we need to allow for more general one-qubit measurements and to perform them in an adaptive fashion as the computation proceeds from the starting point till the end.

To be more specific, let us consider a generic qubit state with complex coefficients

$$|\varphi\rangle_{\mathbf{v}} := c_{\mathbf{v}}^0|0\rangle_{\mathbf{v}} + c_{\mathbf{v}}^1|1\rangle_{\mathbf{v}}, \quad c_{\mathbf{v}}^0, c_{\mathbf{v}}^1 \in \mathbb{C}. \quad (26)$$

Then, a MQC starts with a planar color code (11) and we apply a series of projective measurements $M_{\mathbf{v}} := |x\rangle_{\mathbf{v}}\langle x|$, $x = 0, 1$, from the first qubit $\mathbf{v} = 1$ in the code until the last one $\mathbf{v} = |\mathfrak{V}|$. The order in which this sequence of measurements is carried out through the 2-colex lattice is arbitrary. After each measurement, the corresponding qubit at the vertex \mathbf{v} gets projected onto one of the states in (26) and the result is a value for the outcome denoted by $m_{\mathbf{v}} = 0, 1$.

The result of a run of a MQC is a set of outputs $m_1, \dots, m_{\mathbf{v}}, \dots, m_{|\mathfrak{V}|}$ with a certain probability distribution $P(m_1, \dots, m_{|\mathfrak{V}|})$. We adopt the definition [2] that a MQC is classically simulable in an efficient way if there exists a classical randomized algorithm that allows one to sample the outputs $m_1, \dots, m_{|\mathfrak{V}|}$ from the probability distribution $P(m_1, \dots, m_{|\mathfrak{V}|})$ in a time $\text{poly}(|\mathfrak{V}|)$.

Then, after a complete run of a MQC with (11) we get the following generalized overlapping

$$O_{\text{MQC}} := \langle \Psi_{\mathbf{c}} | \bigotimes_{\mathbf{v} \in \mathfrak{V}} |\varphi\rangle_{\mathbf{v}}. \quad (27)$$

Using the same type of computations that led to (14), we arrive at the following expression

$$\begin{aligned} O_{\text{MQC}} &= \sum_{\gamma \in \Gamma_0} \langle \gamma | \bigotimes_{\mathbf{v} \in \mathfrak{V}} |\varphi\rangle_{\mathbf{v}} \\ &= \prod_{\mathbf{v} \in \mathfrak{V}} c_{\mathbf{v}}^0 \sum_{\gamma \in \Gamma_0} \prod_{\mathbf{v} \in \mathfrak{V} : x_{\mathbf{v}}=1} \left(\frac{c_{\mathbf{v}}^1}{c_{\mathbf{v}}^0} \right). \end{aligned} \quad (28)$$

This result can also be turned into the partition function of a statistical model with 3-body Ising interactions (16) but with complex and inhomogeneous Boltzmann weights

$$w_{ijk} = e^{\beta J_{ijk}} \in \mathbf{C}, \quad (29)$$

depending on each triangular plaquette $\langle i, j, k \rangle$. The generalized overlapping (28) is proportional to a generalized partition function of a 3-body Ising model with inhomogeneous and complex coupling constants

$$\mathcal{Z}(\beta, \{J_{ijk}\}) := \sum_{\{\sigma\}} e^{\beta \sum_{\langle i, j, k \rangle \in \Lambda} J_{ijk} \sigma_i \sigma_j \sigma_k}, \quad (30)$$

where the lattice Λ can be the complete triangular or the complete Union Jack lattice.

At an intermediate stage of a run of a MQC with the color code state, the 2-colex will split into two subsets $\mathcal{C}_2 := \mathfrak{M} \cup \overline{\mathfrak{M}}$, with \mathfrak{M} the set of measured qubits and $\overline{\mathfrak{M}}$ the set of unmeasured qubits [2]. Thus, during the running of the MQC starting with the color code state, we shall find generalized partition functions of the type (30) but for a lattice that is the dual of the subset of measured qubits: $\Lambda = \mathfrak{M}^*$.

Therefore, we arrive at the situation that in order to classically simulate a topological color code state in a MQC we need to simulate the conditional probabilities $P(m_{\mathfrak{v}+1} | m_1, \dots, m_{|\mathfrak{v}|})$ (at step $\mathfrak{v} + 1$ knowing the probabilities of previous steps) and for these we need to be able to compute efficiently in a time $\text{poly}(L)$ on the size L of the intermediate lattice at step $\mathfrak{v} + 1$.

At this point there is a sharp difference between the classical simulation with MQC of Kitaev states and color code states. Kitaev states can be classically simulated under very general conditions: the subsets \mathfrak{M} and $\overline{\mathfrak{M}}$ must be simply connected. The basic ingredient to achieve this result in the 2-body Ising model is that even though a generalized standard Ising model (with arbitrary complex and inhomogeneous couplings) may not be translationally invariant, nevertheless there always exist a technique allowing it to be mapped onto a dimer covering problem (DCP) which in turn can be solved efficiently through the Pfaffian method in polynomial time [2], [23], [24].

However, the dimer problem technique is applicable to 2-body interactions but not for the 3-body interactions that arise in the generalized statistical mechanical models from color codes. In the case of the 3-body Ising model with uniform and real couplings $J_{ijk} = J \in \mathbf{R}$ in a triangular lattice, it can be exactly solved by mapping it onto the generating function of a suitable site-colouring problem (SCP) on a hexagonal lattice [17], [18], which can be solved by the Bethe ansatz. In order for this site-coloring mapping to work, certain restrictive conditions on the triangular lattice must be fulfilled. In particular, the partition function (16) has to be defined on a periodic triangular lattice with L rows in the horizontal direction and N columns in the vertical direction. Let us denote it as $Z_{LN}^{(3)}$. Let us also denote by Z_{MN}^{SCP} the generating

function of a site-coloring problem on a hexagonal lattice with $M = \frac{2}{3}L$ and N columns. Then, the aforementioned mapping works in the limit $N \rightarrow \infty$ as

$$Z_{LN}^{(3)} = Z_{MN}^{\text{SCP}}, \quad N \gg 1. \quad (31)$$

Furthermore, the SCP is solved by Bethe ansatz technique. This technique also poses another fundamental problem in this situation since it is used to compute the eigenvalues of the associated transfer matrix of the SCP and then the issue about the completeness of that spectrum in terms of Bethe ansatz eigenfunctions arises. This issue is always a difficult question and, strictly speaking, it is a conjecture. Quite on the contrary, these difficulties are absent in the standard Ising model case since the DCP is more versatile.

The situation becomes even more difficult if we consider the generalized partition function (30) in the framework of an intermediate step in the MQC. Then, it is not known how to solve it efficiently with a mapping to a SCP in an hexagonal lattice without restrictions.

This site-coloring mapping plays a similar role than the dimer covering mapping in the standard 2-body Ising model. However, the known solutions to this site-coloring problem demand more restrictive conditions on the type of lattices and they are less powerful than the dimer mapping technique.

As for the topological color code on a 2-colex like the Union Jack lattice, similar conclusions apply: in the case of real couplings $J_{ijk} = J \in \mathbf{R}$, it is exactly solvable since it can be mapped onto a 8-vertex model [19], which in turn has to be solved by the Bethe ansatz.

The fact that the 3-body classical Ising model is exactly solvable in the triangular and Union Jack lattices for real and isotropic couplings, is not enough so as to conclude that the corresponding topological color code states are classically simulable in a MQC scenario. Therefore, the classical simulability of topological color codes with MQC remains an open problem.

V. CLUSTER STATES AND MODELS WITH AN EXTERNAL FIELD

In a MQC scheme of quantum computation, the input state is called a cluster state [5] which is a rather general entangled state associated to a great variety of graph states, i.e., states constructed from qubit states located at the vertices of a lattice specified by the incidence matrix of a graph. A very important property of these cluster states is that they can be created efficiently in any system with a quantum Ising-type interaction between two-state particles in the specified lattice configuration. In Sect. IV we have assumed that the input state for the MQC was a TCC state, without caring about how it could be prepared. Here we show how such a topological color code state can be obtained from an appropriate cluster state. As a by-product, this construction will turn out to be useful for obtaining classical Ising models in

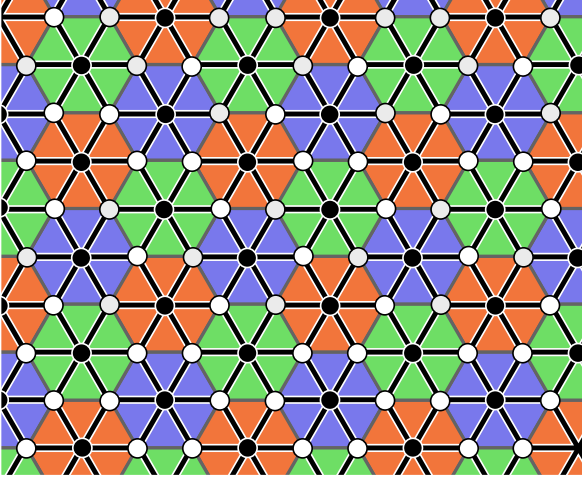


Figure 7: The graph needed to obtain a color code state from a cluster state. The graph is bipartite, and the vertices are divided in black and white. Black vertices correspond to the faces of the 2-colex. White vertices correspond to the vertices of the 2-colex. Black thick lines represent the edges of the graph, and grey lines correspond to the edges of the 2-colex.

an external magnetic field being associated to color code states.

A. Cluster state formulation of TCC

Instead of giving a general definition of cluster states, we will consider only bipartite cluster states. So let \mathfrak{G} be a finite bipartite graph, that is, a graph such that its set of vertices \mathfrak{U} is the disjoint union of two sets $\mathfrak{U} = \mathfrak{U}_1 \cup \mathfrak{U}_2$ in such a way that neighboring vertices never belong to the same \mathfrak{U}_i . Consider the quantum system obtained by attaching a qubit to each of the vertices u . For each such vertex, we denote by $N(u)$ the set containing both u and its neighbors. The cluster state $|\kappa\rangle$ for the graph \mathfrak{G} is then completely characterized by the conditions:

$$\begin{aligned} \forall u \in \mathfrak{U}_1, \quad X_{N(u)}|\kappa\rangle &= |\kappa\rangle, \\ \forall u \in \mathfrak{U}_2, \quad Z_{N(u)}|\kappa\rangle &= |\kappa\rangle. \end{aligned} \quad (32)$$

In order to relate the TCC \mathcal{C} of a 2-colex \mathfrak{C}_2 to a cluster state, we construct a bipartite graph \mathfrak{G} by setting $\mathfrak{U}_1 = \mathfrak{V}$ and $\mathfrak{U}_2 = \mathfrak{F}$. Then the edges are defined so that $u_1 = v \in \mathfrak{U}_1$ is a neighbor of $u_2 = f \in \mathfrak{U}_2$ if v is a vertex of f in \mathfrak{C}_2 , see Fig. 7. Observe that the corresponding cluster state $|\kappa\rangle$ satisfies

$$\forall f \in \mathfrak{F}, \quad X_f|\kappa\rangle = |\kappa\rangle, \quad (33)$$

because $X_f = \bigotimes_{v \in f} X_{N(v)}$. In fact, if γ is a string-net with $\partial_c \gamma = 0$, then $X_\gamma|\kappa\rangle = |\kappa\rangle$. However, to keep things as simple as possible, let us just consider 2-colexes in which all closed string-nets are boundaries. If we measure in the Z basis all the qubits corresponding to vertices

$u \in \mathfrak{U}_2$ we will obtain a series of binary values $x_u = x_f$. The remaining qubits are then in a state characterized by the conditions:

$$\begin{aligned} \forall f \in \mathfrak{F}, \quad X_f|\kappa\rangle &= |\Psi_c(\mathbf{x})\rangle, \\ \forall f \in \mathfrak{F}, \quad Z_f|\kappa\rangle &= x_f|\Psi_c(\mathbf{x})\rangle. \end{aligned} \quad (34)$$

Thus, if $x_f = 0$ for all the faces f , the result is the TCC state $|\Psi_c\rangle$ (11). In the other cases the result is essentially a TCC, in particular the state is

$$|\Psi_c(\mathbf{x})\rangle := \sum_{\gamma \in \Gamma_{\mathbf{x}}} |\gamma\rangle \quad (35)$$

where \mathbf{x} denotes the binary vector of the measurement results and $\Gamma_{\mathbf{x}}$ is the set of string-nets γ with $\partial_c \gamma = \sum_f x_f f$.

In fact, the original cluster state can be written as follows:

$$|\kappa\rangle = \sum_{\mathbf{x}} \left(|\mathbf{x}\rangle \otimes \sum_{\gamma \in \Gamma_{\mathbf{x}}} |\gamma\rangle \right) \quad (36)$$

where $|\mathbf{x}\rangle = \bigotimes_{u \in \mathfrak{U}_2} |x_u\rangle_u$ is an element of the computational basis of the subsystem of qubits in \mathfrak{U}_2 . To check that this is indeed the correct expression for the cluster state, it is enough to note that it satisfies (32).

B. Models with an external field

Thus far, we have only considered statistical mechanical models with zero external magnetic field. Here we go beyond that situation, considering the formulation of models with 3-body Ising interactions and arbitrary magnetic fields from the projection of topological color codes onto appropriate product states.

Let us define the product state

$$|\Phi_P(\mathbf{J}, \mathbf{h})\rangle := \bigotimes_{v \in \mathfrak{U}_1} |\phi(J_v)\rangle_v \bigotimes_{f \in \mathfrak{U}_2} |\phi(h_f)\rangle_f, \quad (37)$$

where $\mathbf{J} = (J_v) \in \mathbf{R}^{|\mathfrak{V}|}$, $\mathbf{h} = (h_f) \in \mathbf{R}^{|\mathfrak{F}|}$ and

$$|\phi(s)\rangle := \cosh s |0\rangle + \sinh s |1\rangle, \quad s \in \mathbf{R}. \quad (38)$$

Consider the overlapping

$$O(\beta, \mathbf{J}, \mathbf{h}) := \langle \Psi_c | \Phi_P(\beta \mathbf{J}, \beta \mathbf{h}) \rangle. \quad (39)$$

Its value is

$$\begin{aligned} \sum_{\mathbf{x}} \langle \mathbf{x} | \bigotimes_{f \in \mathfrak{U}_2} |\phi(h_f)\rangle_f \sum_{\gamma \in \Gamma_{\mathbf{x}}} \langle \gamma | \bigotimes_{v \in \mathfrak{U}_1} |\phi(J_v)\rangle_v = \\ \prod_f \cosh(\beta h_f) \prod_v \cosh(\beta J_v) \sum_{\mathbf{x}} \sum_{\gamma \in \Gamma_{\mathbf{x}}} \prod_f u_f^{x_f} \prod_v u_v^{\gamma_v}, \end{aligned} \quad (40)$$

where

$$u_f := \tanh(\beta h_f), \quad u_v := \tanh(\beta J_v). \quad (41)$$

We want to relate (39) to the partition function of a classical spin system. As in section III, we consider the lattice Λ dual to the 2-colex \mathfrak{C}_2 and we associate a classical system to Λ by attaching classical spin variables $\sigma_i = \pm 1$ to each of its sites i . This time however we include triangle dependent couplings J_{ijk} in triangles and a site dependent external field h_i . Thus, we want to derive the high temperature expansion for the partition function

$$\mathcal{Z}(\beta, \mathbf{J}, \mathbf{h}) := \sum_{\{\sigma\}} e^{-\beta \mathcal{H}(\mathbf{J}, \mathbf{h})}, \quad (42)$$

where the classical Hamiltonian is

$$\mathcal{H}(\mathbf{J}, \mathbf{h}) := - \sum_i h_i \sigma_i - \sum_{\langle i, j, k \rangle} J_{ijk} \sigma_i \sigma_j \sigma_k. \quad (43)$$

We start using the identities

$$\begin{aligned} e^{\beta h_i \sigma_i} &= \cosh(\beta h_i) + \sigma_i \sinh(\beta h_i), \\ e^{\beta J_{ijk} \sigma_i \sigma_j \sigma_k} &= \cosh(\beta J_{ijk}) + \sigma_i \sigma_j \sigma_k \sinh(\beta J_{ijk}), \end{aligned} \quad (44)$$

so that,

$$\mathcal{Z}(\beta, \mathbf{J}, \mathbf{h}) = C \sum_{\{\sigma\}} \prod_i (1 + u_i \sigma_i) \prod_{\langle i, j, k \rangle} (1 + u_{ijk} \sigma_i \sigma_j \sigma_k), \quad (45)$$

where

$$C := \prod_i \cosh(\beta h_i) \prod_{\langle i, j, k \rangle} \cosh(\beta J_{ijk}), \quad (46)$$

$$u_i := \tanh(\beta h_i), \quad u_{ijk} := \tanh(\beta J_{ijk}). \quad (47)$$

Let us rewrite (45) in the form

$$\mathcal{Z}(\beta, \mathbf{J}, \mathbf{h}) = C \sum_{\mathbf{x}} \sum_{\delta} \sum_{\{\sigma\}} \prod_i (u_i \sigma_i)^{x_i} \prod_{\langle i, j, k \rangle} (u_{ijk} \sigma_i \sigma_j \sigma_k)^{\delta_{ijk}}, \quad (48)$$

where $\mathbf{x} = (x_i)$ is a binary vector and $\delta = \sum_{\langle i, j, k \rangle} \delta_{ijk} \Delta_{ijk}$ is a chain of triangles, that is, a formal sum over triangles with binary coefficients. Reordering the expression we get

$$\mathcal{Z}(\beta, \mathbf{J}, \mathbf{h}) = C \sum_{\mathbf{x}} \sum_{\delta} \epsilon(\mathbf{x}, \delta) \prod_i u_i^{x_i} \prod_{\langle i, j, k \rangle} u_{ijk}^{\delta_{ijk}}, \quad (49)$$

where

$$\epsilon(\mathbf{x}, \delta) := \sum_{\{\sigma\}} \prod_i \sigma_i^{x_i} \prod_{\langle i, j, k \rangle} (\sigma_i \sigma_j \sigma_k)^{\delta_{ijk}}, \quad (50)$$

From (21) it follows that

$$\epsilon(\mathbf{x}, \delta) = \begin{cases} 2^N, & \text{if } \forall i \prod_{\langle j, k \rangle_i} = x_i, \\ 0, & \text{in other case.} \end{cases} \quad (51)$$

where $\langle j, k \rangle_i$ denotes the pairs (j, k) which form a triangle with i , and N is the number of sites. Finally we can give the desired high temperature expansion of the partition function:

$$\mathcal{Z}(\beta, \mathbf{J}, \mathbf{h}) = 2^N C \sum_{\mathbf{x}} \sum_{\delta \in \Delta_{\mathbf{x}}} \prod_i u_i^{x_i} \prod_{\langle i, j, k \rangle} u_{ijk}^{\delta_{ijk}}, \quad (52)$$

where $\Delta_{\mathbf{x}}$ contains those chains of triangles such that at any given site i an even (odd) number of triangles meet if $x_i = 0$ ($x_i = 1$).

In order to compare (40) and (52), we first observe that sites i correspond to faces of the 2-colex $\mathfrak{f} \in \mathfrak{F} = \mathfrak{U}_2$ and triangles Δ_{ijk} correspond to vertices of the 2-colex $\mathfrak{v} \in \mathfrak{V} = \mathfrak{U}_1$. This correspondence relates in an obvious way x_i with x_v , h_i with h_v and so on and so forth. Also, there is an exact correspondence between chains of triangle δ and string-nets γ . In particular, this correspondence identifies $\Delta_{\mathbf{x}}$ with $\Gamma_{\mathbf{x}}$, so that we get the desired relationship between the overlapping and the partition function

$$\mathcal{Z}(\beta, \mathbf{J}, \mathbf{h}) = 2^N O(\beta, \mathbf{J}, \mathbf{h}). \quad (53)$$

VI. CONCLUSIONS

We have shown that the classical spin models associated to quantum topological color code states in the two dimensional lattices called 2-colexes are Ising models with 3-body interactions. We have studied this mapping in the triangular and the Union Jack lattices, which are the duals of the two very representative 2-colexes, namely, the hexagonal and the square-octogonal lattices, respectively. This is a genuine difference with respect to the case with toric code states which yield the partition function of the standard Ising model with two-body interactions. Ising models with different n-body interactions have very different properties in general. Remarkably, different computational capabilities of the topological color codes depending on the chosen 2-colex correspond to different universality classes of the associated classical 3-body Ising models.

The tools employed to relate classical spin models with topological color code states can be extended to study the performance of such topological states when they are considered as input in a measurement-based quantum computation. Then, the classical 3-body models that arise involve arbitrary complex couplings and lattice shapes. The problem of evaluating their corresponding generalized partition functions cannot be performed with the dimer covering technique that is so successful in the case of the classical 2-body Ising model. The similar technique in the 3-body case is a particular site-coloring problem that only in very specific instances has been solved by means of the Bethe ansatz. The completeness of the Bethe ansatz poses in turn more fundamental problems in this regard. Therefore, the fact

that the 3-body Ising model is exactly solvable in certain conditions is not enough to conclude so far that the parent quantum topological color code states are efficiently classically simulated by MQC.

Another interesting result is the construction of a cluster state from which we can construct the topological color code state. This turns out to be useful in order to obtain classical spin models in the presence of arbitrary external magnetic fields. Likewise, there are other two-

dimensional multipartite states that arise in the study of quantum antiferromagnets that may lead to a variety of classical spin models [25].

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